A Sound and Complete Diagrammatic Reasoning System

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Abstract
Simple spider diagrams are a system of Venn-type diagrams that can be used to reason diagrammatically about sets, their cardinalities and their relationships. They extend the systems of Venn-Peirce diagrams investigated by Shin to include lower bounds for the cardinalities of the sets represented by the diagrams. This paper summarises the main syntax and semantics of simple spider diagrams and introduces inference rules for reasoning with the system. We discuss the soundness of the system and our principal result is a completeness theorem for simple spider diagrams. In proving completeness, we develop a proof strategy that is simpler than that adopted by Shin. We expect this strategy to extend to other, richer diagrammatic systems including spider diagrams and the constraint diagram systems that have been used in conjunction with object-oriented modelling notations such as the Unified Modelling Language.

Keywords Diagrammatic reasoning, visual formalisms.

1. Introduction
In object-oriented software development, diagrammatic modelling notations are used to specify systems. Recently, the Unified Modelling Language (UML) [11] has become the Object Management Group’s (OMG) standard for such notations. In UML, constraints, such as invariants, preconditions and postconditions, are expressed using the Object Constraint Language (OCL) [14], essentially a stylised, textual, form of first-order predicate logic, which is part of the UML standard. Constraint diagrams [4, 9] are a diagrammatic notation for expressing constraints and can be used in conjunction with UML and OCL. Spider diagrams [3, 8] emerged from work on constraint diagrams. They combine and extend Venn diagrams and Euler circles to express constraints on sets and their relationships with other sets.

The notation we now call Euler circles was introduced to illustrate relations between classes [2]. This notation uses the topological properties of enclosure, exclusion and intersection to represent the set-theoretic notions of subset, disjoint sets, and set intersection, respectively. In 1880, Venn modified this notation to represent logical propositions [13]. In Venn diagrams, all possible intersections of the closed curves must be shown and shading is used to show that a particular region represents the empty set. In the 1890s, Peirce modified Venn diagrams by including X-sequences to introduce elements and disjunctive information into the system [10]. Recently, full formal semantics and inference rules have been developed for Venn-Peirce diagrams [12] and Euler diagrams [6]; see also [1, 5] for related work. Shin proves soundness and completeness results for two systems of Venn-Peirce diagrams.

This paper extends these diagrammatic inference rules to simple spider diagrams. Simple spider diagrams, in effect, enhance the semantics of the second Venn-Pierce system that Shin investigated (i.e., Venn-II, see [12] Chapter 4) to give lower bounds for the cardinality of the sets represented by the diagrams. Shin’s proof of completeness does not extend very easily to this system, or to any other spider diagram or constraint diagram system; the central notion of a maximal diagram is not easy to define for these systems. In proving completeness of the simple spider diagram system, we give a proof strategy that should be extensible to most spider/constraint diagram systems and other similar systems, and can be adapted to give a simpler proof of the completeness of the Venn-II system than the one given by Shin. The general aim of this work to provide the necessary mathematical underpinning for the development of software tools to aid the reasoning process.

A discussion of simple spider diagrams is conducted in section 2, where the main syntax and semantics of the notation is introduced. Section 3 introduces the inference rules for reasoning with simple spider diagrams and for combining diagrams. Section 4 considers the validity of the inference rules culminating in the soundness theorem. Section 5 gives the strategy for proving completeness and proves the completeness theorem. Section 6 states the conclusions of this paper and details related, ongoing and future work.
Throughout this extended abstract, for space reasons, some details of proofs, and sometimes whole proofs, will be omitted; however, sufficient information will be given to judge the veracity of the approach.

2. Simple spider diagrams

This section introduces the main syntax and semantics of simple spider diagrams. Simple spider diagrams are Venn-Peirce diagrams adapted so that we can infer lower bounds for the cardinalities of the sets represented by the non-empty regions. Spider diagrams contain other syntactic elements which enable the expression of relations between elements and are based on Euler diagrams rather than Venn diagrams; they form the basis of constraint diagrams, a rich notation allowing relations between sets to be expressed, see [3, 4, 8, 9] for more details.

2.1. Syntactic elements of unitary simple spider diagrams

A contour is a simple closed plane curve. A boundary rectangle properly contains all other contours. A district (or basic region) is the bounded region of the plane enclosed by a contour. A region is defined, recursively, as follows: any district is a region; if \( r_1 \) and \( r_2 \) are regions, then the union, intersection, or difference, of \( r_1 \) and \( r_2 \) are regions provided these are non-empty. A zone (or minimal region) is a region having no other region contained within it. Contours and regions denote sets.

A spider is a tree with nodes (called feet) placed in different zones; the connecting edges (called legs) are straight lines. A spider touches a zone if one of its feet appears in that region. A spider may touch a zone at most once. A spider is said to inhabit the region which is the union of the zones it touches. For any spider \( s \), the habitat of \( s \), denoted \( \eta(s) \), is the region inhabited by \( s \). The set of complete spiders within region \( r \) is denoted by \( S(r) \). A spider denotes the existence of an element in the set denoted by the habitat of the spider. Two distinct spiders denote distinct elements.

Every region is a union of zones. A region is shaded if each of its component zones is shaded. A shaded region denotes the empty set. No spider’s foot can touch a shaded region. A unitary simple spider diagram is a boundary rectangle together with a finite collection of contours (all possible intersections of contours must occur, i.e., the underlying diagram is a Venn diagram), spiders and shaded regions. Each contour must be labelled and no two contours in the same unitary diagram can have the same label. The labelling of spiders is optional. For any unitary simple spider diagram \( D \), we use \( C = C(D), Z = Z(D), Z^* = Z^*(D), R = R(D), R^* = R^*(D), L = L(D) \) and \( S = S(D) \) to denote the sets of contours, zones, shaded zones, regions, shaded regions, contour labels and spiders of \( D \), respectively.

![Figure 1](image-url)

The diagram \( D \) in Figure 1 can be interpreted as:

\[
A - (B \cup C) = \emptyset \land C - (A \cup B) = \emptyset \land A \land C - B \geq 2 \land \exists x, y \bullet x \in B \land y \in B - C \land x \neq y.
\]

2.2. Semantics of unitary simple spider diagrams

A model for a unitary simple spider diagram \( D \) is a pair \( m = (\Psi, U) \) where \( U \) is a set and \( \Psi : C \to Set U \), where \( Set U \) denotes the power set of \( U \), is a function mapping contours to subsets of \( U \). The boundary rectangle is interpreted as \( U \).

A zone is uniquely defined by the contours containing it and the contours not containing it; its interpretation is the intersection of the sets denoted by the contours containing it and the complements of the sets denoted by those contours not containing it. We extend the domain of \( \Psi \) to interpret regions as subsets of \( U \). First define \( \Psi : Z \to Set U \) by

\[
\Psi(z) = \bigcap_{c \in C^*} \Psi(c) \cap \bigcap_{c \in C^c} \Psi(c)
\]
where $C^+(z)$ is the set of contours containing the zone $z$, $C^-(z)$ is the set of contours not containing $z$ and $\overline{\Psi}(c) = U - \Psi(c)$, the complement of $\Psi(c)$. Since any region is a union of zones, we may define $\Psi: R \to \text{Set } U$ by

$$\Psi(r) = \bigcup_{z \in Z(r)} \Psi(z)$$

where, for any region $r$, $Z(r)$ is the set of zones contained in $r$.

The semantics predicate $P_D(m)$ of a unitary diagram $D$ is the conjunction of the following conditions.

**Spider Condition**: A spider denotes the existence of an element in the set denoted by the habitat of the spider:

$$\bigwedge_{s \in S} \exists x \cdot x \in \Psi(\eta(s))$$

**Distinct Spiders Condition**: The elements denoted by two distinct spiders are distinct:

$$\bigwedge_{s, t \in S, s \neq t} \exists x \cdot x \neq x$$

**Shading Condition**: The set denoted by a shaded zone is empty:

$$\bigwedge_{z \in Z} \Psi(z) = \emptyset$$

**Theorem 1** The cardinality of the set denoted by region $r$ of unitary diagram $D$ is greater than or equal to the number of complete spiders in $r$:

$$\forall r \in R(D) \cdot |\Psi(r)| \geq |S(r)|$$

Theorem 1 is equivalent to the conjunction of the Spider and Distinct Spider Conditions. The proof is omitted.

### 2.3. Compound diagrams and multi-diagrams

Given two unitary diagrams $D_1$ and $D_2$, we can connect $D_1$ and $D_2$ with a straight line to produce a diagram $D = D_1 - D_2$. If a diagram has one boundary rectangle, then it is a unitary diagram; if a diagram has more than one rectangle, then it is a compound diagram. If a compound diagram $D$ has $n$ components, then we can place those $n$ components in any order. That is, $D_1 - D_2 - \ldots - D_n = D_n - D_{n-1}$.

The semantics predicate of a compound diagram $D$ is the disjunction of the semantics predicates of its component unitary diagrams; the boundary rectangles of the component unitary diagrams are interpreted as the same set $U$. That is,

$$P_D(m) = \bigvee_{i=1}^n P_{D_i}(m)$$

where $D = D_1 - D_2 - \ldots - D_n$.

Contours with the same labels in different component unitary diagrams of a compound diagram $D$ are interpreted as the same set.

$$\forall c_1, c_2 \in C(D) \cdot (c_1) = (c_2) \cdot \Psi(c_1) = \Psi(c_2)$$

where $(c)$ is the label of contour $c$.

![Figure 2](image-url)
The compound diagram $D$ in Figure 2 asserts that: $(\exists x, y \cdot x \in A \cap C \land y \in B - C) \lor (\exists x, y \cdot x \in B \land y \in A - B \cup C)$.

A simple spider multi-diagram is a finite collection $\Delta$ of simple spider diagrams. The semantics predicate of a multi-diagram is the conjunction of the semantics predicates of the individual diagrams; the boundary rectangles of all diagrams are interpreted as the same set $U$. That is,

$$P_\Delta(\Psi) = \bigwedge_{D \in \Delta} P_D(\Psi).$$

Contours with the same labels in different individual diagrams of a multi-diagram $\Delta$ are interpreted as the same set:

$$\forall c_1, c_2 \in \bigcup_{D \in \Delta} C(D) \cdot (c_1) = (c_2) \quad \Psi(c_1) = \Psi(c_2).$$

### 2.4 Comparing regions across diagrams

Let $D$ be a unitary diagram. For any $z \in Z(D)$, define $L^+(z) = (C^+(z))$, the set of labels of the contours containing $z$, and $L^-(z) = (C^-(z))$.

Given two unitary diagrams $D$ and $D'$, we can define equivalent regions across the two diagrams by considering partitions of the set of contour labels the two diagrams have in common. Let $P = (L^+, L^-)$ be a partition of $L(D) \cap L(D')$ and define $Z_P(D) = \{z \in Z(D) \mid L^+ = L^+(z) \cap L(D') \land L^- = L^-(z) \cap L(D')\}$. A region $zr \in R(D)$ is said to be zonal with respect to $D'$ if there exists a partition $P$ of $L(D) \cap L(D')$ such that

$$zr = \bigcup_{z \in Z_P(D)} z.$$

Suppose region $zr$ of $D$ is zonal with respect to $D'$ and $zr'$ of $D'$ is zonal with respect to $D$. Then $zr$ and $zr'$ are corresponding zonal regions, denoted $zr \equiv_c zr'$, if there exists a partition $P$ of $L(D) \cap L(D')$ such that

$$zr = \bigcup_{z \in Z_P(D)} z \quad \text{and} \quad zr' = \bigcup_{z \in Z_P(D')} z'.$$

Let $r$ be a region of $D$ and let $r'$ be a region of $D'$. Then $r$ and $r'$ are corresponding regions, denoted by $r' \equiv_c r$, if and only if $r$ is a union of a set $ZR(r)$ of zonal regions with respect to $D'$, $r'$ is a union of a set $ZR(r')$ of zonal regions with respect to $D$, and

$$\forall zr \in ZR(r) \exists zr' \in ZR(r') \cdot zr \equiv_c zr' \land \forall zr' \in ZR(r') \exists zr \in ZR(r) \cdot zr \equiv_c zr'.$$

**Theorem 2** The relation $\equiv_c$ is an equivalence relation.

If $r_1 \in R(D)$ and $r_1 \subseteq r \equiv_c r'$, then $r_1$ is a corresponding subregion of $r'$, denoted by $r_1 \subseteq_c r'$.

In Figure 3, the region $z = z_1 \cup z_2$ in $D$ is zonal with respect to $D'$ and the region $z'$ in $D'$ is zonal with respect to $D$. Furthermore, $z' \equiv_c z$ as both regions are associated with the partition $P = (\{B\}, \{A\})$ of $L(D) \cap L(D') = \{A, B\}$; hence $z_1 \subseteq_c z'$ and $z_2 \subseteq_c z'$. 

**Figure 3**
Theorem 3  
(i) $\forall r \in R(D) \forall r' \in R(D') \forall m = (\Psi, U) \bullet P_D(m) \land P_{D'}(m) \bullet r \equiv r' \quad \Psi(r) = \Psi(r')$.

(ii) $\forall r \in R(D) \forall r' \in R(D') \forall m = (\Psi, U) \bullet P_D(m) \land P_{D'}(m) \bullet r \subseteq r' \quad \Psi(r) \subseteq \Psi(r')$.

The proof is omitted. We can now give a definition of equivalent diagrams. Two unitary diagrams $D$ and $D'$ are equivalent, denoted by $D \equiv D'$, if

(i) $L(D) = L(D')$,

(ii) $\forall r \in R^*(D) \exists r' \in R^*(D') \bullet r \equiv r' \land \forall r' \in R^*(D') \exists r \in R^*(D) \bullet r \equiv r'$ and

(iii) $\forall r \in R(D) \forall r' \in R'(D') \bullet r \equiv r' \quad |S(r)| = |S(r')|$.  

2.5. Compliance and Consistency

A model $m = (\Psi, U)$ complies with diagram $D$ if it satisfies its semantic predicate $P_D(m)$. We write $m \models D$. That is,

$m \models D \iff P_D(m)$.

Similarly, a model $m$ complies with multi-diagram $\Delta$ if it satisfies its semantic predicate $P_\Delta(m)$. That is,

$m \models \Delta \iff P_\Delta(m)$.

A diagram is consistent iff it has a compliant model. Similarly, a multi-diagram is consistent iff it has a compliant model.

Theorem 4  All simple spider diagrams are consistent.

The proof is based on the construction of topological models for the diagram. The details are omitted. Theorem 4 does not extend to multi-diagrams.

Theorem 5  (i) Two unitary simple spider diagrams $D$ and $D'$ are consistent iff

$\forall r \in R(D) \forall r' \in R(D') \bullet r \equiv r' \quad \neg(r \in R^*(D) \land |S(r')| > 0) \land \neg(r' \in R^*(D') \land |S(r)| > 0)$.

(ii) Two simple spider diagrams $D$ and $D'$ are consistent iff there exist unitary components $D_i$ of $D$ and $D_j$ of $D'$ such that $D_i$ and $D_j$ are consistent.

The formal details of the proof are again omitted. Intuitively, the diagrammatic condition in (i) would prevent the case in which a region is shaded in one diagram but the corresponding region in the other contains a spider; this is the only case in which a pair of diagrams can be inconsistent.

Figure 4 shows a multi-diagram which is inconsistent, but whose components are all pairwise consistent. Discussion of the consistency of multi-diagrams in general is deferred until we consider combining diagrams.
3. Diagrammatic reasoning rules

We introduce purely syntactic, diagrammatic rules for turning one diagram into another. In this section we define and illustrate the rules; in the next section we show that the rules are valid.

3.1. Rules of transformation of unitary diagrams

We introduce rules that allow us to obtain one unitary diagram from a given unitary diagram by removing, adding or modifying diagrammatic elements. In several cases the statement of the rule is self-explanatory.

Rule 1: Erasure of shading. We may erase the shading in an entire zone.

Rule 2: Erasure of a spider. We may erase a complete spider.

Rule 3: Erasure of a contour. We may erase a contour. When a contour is erased:
• any shading remaining in only a part of a zone should also be erased.
• if a spider has feet in two regions which combine to form a single zone with the erasure of the contour, then these feet are replaced with a single foot connected to the rest of the spider.

Rule 3 is illustrated in Figure 5.

Rule 4: Spreading the feet of a spider. If a diagram has a spider \( s \), then we may draw a node in any non-shaded zone \( z \) which does not contain a foot of \( s \) and connect it to \( s \).

Rule 4 is illustrated in Figure 6. From \( D \) we know that there is an element belonging to \( A - B \). Having spread its feet in \( D' \), we may only infer that this element belongs to \( A \).

Rule 5: Introduction of a contour. A new contour may be drawn interior to the bounding rectangle observing the partial-overlapping rule: each zone splits into two zones with the introduction of the new contour. Each foot of a spider is replaced with a connected pair of feet, one in each new zone.

In Figure 7, a new contour \( C \) is introduced satisfying the partial overlapping rule. Each zone in \( D \) becomes a pair of zones in \( D' \) and each foot of the spider bifurcates to become two feet, one in each new zone.
3.2. Rules of transformation of involving compound diagrams

Rule 6: Splitting spiders. If a unitary diagram $D$ has a spider $s$ whose habitat is formed by $n$ zones, then we may replace $D$ with a connection of $n$ unitary diagrams $D_1$–…–$D_n$ where each foot of the spider $s$ touches a different corresponding zone in each diagram $D_i$.

Rule 6 is illustrated in Figure 8.

Rule 7: Rule of excluded middle. If a unitary diagram $D$ has a non-shaded zone $z$ touched by no spiders, then we may replace $D$ with $D_1$–$D_2$, where $D_1$ and $D_2$ are unitary and one of the corresponding zones of $z$ is shaded and the other touched by a single foot spider.

Rule 7 is illustrated in Figure 9.

Rule 8: The rule of connecting a diagram. For a given diagram $D$, we may connect any diagram $D'$ to $D$.

Rule 9: The rule of construction. Given a diagram $D_1$–…–$D_n$, we may transform it into $D$ if each $D_1$,…, $D_n$ may be transformed into $D$ by some of the first eight transformation rules.

Rule 10: The rule of inconsistency. Given an inconsistent multi-diagram $\Delta$, we may replace $\Delta$ with any multi-diagram.
3.3 Combining Diagrams

Given two consistent diagrams, \( D^1 \) and \( D^2 \), we can combine them to produce a diagram \( D \), losing no semantic information in the process. In this section we describe the construction of such a combined diagram \( D \).

An \( \alpha \) diagram is a diagram in which no spider’s legs appear, that is, the habitat of any spider is a zone. Any diagram \( D \) can be transformed into an \( \alpha \) diagram by repeated application of rule 6, splitting spiders. We give the rule for combining diagrams in several stages.

**Rule 11: The rule of combining diagrams**

Let \( D^1 \) and \( D^2 \) be two diagrams. Then their combination \( D = D^1 \star D^2 \) is defined as follows.

(i) \( D^1 \) and \( D^2 \) are consistent \( \alpha \) unitary diagrams with \( L(D^1) = L(D^2) \). The combined diagram \( D \) is also an \( \alpha \) unitary diagram for which \( L(D) = L(D^1) = L(D^2) \). So, for each \( z \in Z(D) \), there exist equivalent zones \( z_1 \in Z(D^1) \) and \( z_2 \in Z(D^2) \). Furthermore, the number of spiders in \( z \) is equal to the maximum of the number of spiders in \( z_1 \) and the number of spiders in \( z_2 \), and \( z \) is shaded iff \( z_1 \) or \( z_2 \) is shaded.

\[
\forall z \in Z(D) \exists z_1 \in Z(D^1) \exists z_2 \in Z(D^2) \bullet z \equiv_z z_1 \equiv_z z_2
\]

\[
(S(z) \equiv \max(|S(z_1)|, |S(z_2)|) \wedge (z \in Z^*(D) \iff z_1 \in Z^*(D^1) \lor z_2 \in Z^*(D^2)))
\]

(ii) \( D^1 \) and \( D^2 \) are consistent unitary diagrams and \( L(D^1) \neq L(D^2) \). We introduce contours into \( D^1 \) and \( D^2 \) to obtain \( D^1a \) and \( D^2b \), where \( L(D^1a) = L(D^2b) = L(D^1) \cup L(D^2) \). Transform \( D^1a \) and \( D^2b \) into their \( \alpha \) diagrams \( D^1a_1 - \cdots - D^1a_m \) and \( D^1b_1 - \cdots - D^1b_m \). The combined diagram \( D \) is the compound diagram formed by combining each \( D_i^1a \) with each \( D_j^2b \), where the two components are inconsistent, we do not obtain a corresponding component in \( D \).

(iii) \( D^1 \) and \( D^2 \) are any consistent simple spider diagrams. The combined diagram \( D \) is the compound diagram formed by combining each component \( D^1_i \) of \( D^1 \) with each component \( D^2_j \) of \( D^2 \).

![Figure 10](image-url)
Rule 11(ii) is illustrated in Figure 10. First, contour C is added to $D^2$ to form $D^{2a}$, which is then transformed into an $\alpha$ diagram. We then combine the components of each diagram. Note that $D^{ia}$ and $D^{2a}_2$ are inconsistent, as are $D^{ia}_1$ and $D^{2a}_3$, so the resulting combined diagram is unitary.

**Theorem 6** The combining operation $*$ is (i) commutative, (ii) associative.

The associativity of $*$ allows us to define the combination of the components of a multi-diagram $\Delta = \{D^1, D^2, \ldots, D^n\}$ unambiguously as $D^\ast = D^1 \ast D^2 \ast \cdots \ast D^n$. If $\Delta$ is inconsistent, the result will be no diagram; $D^\ast$ is only defined when $\Delta$ is consistent. A test for the consistency of $\Delta$ is to try to evaluate $D^\ast$.

**4. Soundness**

We write $D \vdash D'$ to denote that a diagram $D'$ is obtained from another diagram $D$ by a single application of a rule. $D'$ is a consequence of $D$, denoted by $D \models D'$, if every compliant model for $D$ is also a compliant model for $D'$. A rule is valid, if $D \vdash D'$, $D \models D'$. We write $D \vdash D'$ to denote that diagram $D'$ is obtained from multi-diagram $\Delta$ by applying a sequence of transformations. We write $D \vdash D'$ to mean $\{D\} \vdash D'$, etc.

For space reasons, we omit the proofs of the validity of rules 1 to 10. These rules are similar to those of the Venn-II system given in [12] and the proofs are fairly straightforward; full formal proofs of these rules can be found in [7]. It can be noted that rules 5, introduction of a contour, 6, splitting spiders, 7, rule of excluded middle, and 11, combining diagrams do not lose any semantic information; this fact is useful for proving completeness (see next section).

**Theorem 1** Rule 11, the rule of combining diagrams, is valid. That is, $D = D^1 \ast D^2 = \{D^1, D^2\} \models D$.

**Proof**

(i) Let $D^1$ and $D^2$ be two consistent $\alpha$ unitary diagrams with $L(D^1) = L(D^2)$. By definition, the combined diagram $D$ is also an $\alpha$ unitary diagram for which $L(D) = L(D^1) = L(D^2)$ and for each $z \in Z(D)$, there exist corresponding equivalent zones $z_1 \in Z(D^1)$ and $z_2 \in Z(D^2)$ for which the number of spiders in $z$ is equal to the maximum of the number of spiders in $z_1$ and the number of spiders in $z_2$, and $z$ is shaded if $z_1$ or $z_2$ is shaded.

Let $m = (\Psi, U)$ be any model for which $m \models \{D^1, D^2\}$. Then $m \models D^1$ and $m \models D^2$. Let $z \in Z(D)$ have corresponding equivalent zones $z_1 \in Z(D^1)$ and $z_2 \in Z(D^2)$.

(a) Assume that $z$ is shaded. Then corresponding zone $z_1$ is shaded or corresponding zone $z_2$ is shaded. So, $\Psi(z_1) = \Psi(z_2) = \emptyset$ and thus $\Psi(z) = \emptyset$.

(b) Assume that $z$ contains $n$ spiders. That is, $|Z(z)| = n$. Then one of the corresponding zones $z_1$ or $z_2$ contains $n$ spiders and the other contains $n$ or fewer. So, $|\Psi(z_1)| = |\Psi(z_2)| \geq n$ and hence $|\Psi(z)| \geq n$.

Therefore $m \models D$. Hence, $\{D^1, D^2\} \models D$.

(ii) Now, let $D^1$ and $D^2$ be two consistent unitary diagrams with $L(D^1) \neq L(D^2)$. We use the notation introduced in rule 11. By repeated application of rule 5, introduction of a contour, $D^1 \vdash D^{ia}_1$ and $D^2 \vdash D^{2b}_1$. Hence, by the validity of rule 5, $D^1 \vdash D^{ia}_1$ and $D^2 \vdash D^{2b}_1$. By repeated application of rule 6, splitting spiders, $D^{ia}_1 \vdash D^{ia}_1 \ast \cdots \ast D^{ia}_m$ and $D^{2b}_1 \vdash D^{2b}_1 \ast \cdots \ast D^{2b}_m$. Hence, by the validity of rule 6, $D^{ia}_1 \vdash D^{ia}_1 \ast \cdots \ast D^{ia}_m$ and $D^{2b}_1 \vdash D^{2b}_1 \ast \cdots \ast D^{2b}_m$.

Then $\{D^{ia}_1 \ast \cdots \ast D^{ia}_m, D^{2b}_1 \ast \cdots \ast D^{2b}_m\} \models D$, where each component $D_j$ of $D$ is obtained by $D_j = D^{ia}_j \ast D^{2b}_j$. Now each $D^{ia}_i$ and each $D^{2b}_j$ is an $\alpha$ diagram, so, by part (i) of this proof, $\{D^{ia}_1, D^{2b}_1\} \models D$. Since $D$ is formed by connecting each of its components by rule 8, connecting a diagram. So, by the validity of rule 8, $\{D^{ia}_1 \ast \cdots \ast D^{ia}_m, D^{2b}_1 \ast \cdots \ast D^{2b}_m\} \models D$. Hence, by transitivity, $\{D^1, D^2\} \models D$.

(iii) Let $D^1$ and $D^2$ be any consistent diagrams. The combined diagram $D$ is the compound diagram formed by combining each component $D^1_i$ of $D^1$ with each component $D^2_j$ of $D^2$. The proof of this follows similarly to the proof of part (ii).
Theorem 1  Soundness Theorem  Let \( \Delta \) be a multi-diagram and \( D' \) a diagram. Then  \( \Delta \vdash D' \quad \Delta \models D' \).

The result follows by induction from the validity of the rules.

5. Completeness

To prove completeness we show that if diagram \( D' \) is a consequence of multi-diagram \( \Delta \), then \( \Delta \) can be transformed into \( D' \) by a finite sequence of applications of the rules given in section 3. That is, \( \Delta \models D' \quad \Delta \vdash D' \).

The basic strategy is to transform \( \Delta \) and \( D' \) into diagrams \( D \) and \( D'' \), respectively, so that if \( D'' \) is a consequence of diagram \( D \), then there are diagrammatic conditions which must hold between \( D \) and \( D'' \), and if these conditions hold, then \( D \) can be transformed syntactically into \( D'' \). We first show that a multi-diagram is a consequence of the combination of its component diagrams. That is, if \( \Delta = \{D^1, D^2, \ldots, D^n\} \), then any model \( m \) compliant with \( D^* = D^1 \ast D^2 \ast \cdots \ast D^n \) is also compliant with \( \Delta \), that is, \( D^* \models \Delta \).

The proofs of Theorems 9, 10, 11 are very informal. The full proofs of all the theorems will appear in [7].

**Theorem 9**  Let \( \Delta = \{D^1, D^2, \ldots, D^n\} \) and \( D^* = D^1 \ast D^2 \ast \cdots \ast D^n \). Then \( D^* \models \Delta \).

**Proof**  Each step in obtaining \( D^* \) is reversible. Thus for each \( i \), \( D^* \vdash D^i \). So, by soundness, \( D^* \models \Delta \).

**Theorem 10**  Let \( D' \) be the diagram obtained from \( D \) by rule 5, introduction of a contour. Then \( D' \models D \).

**Proof**  Introducing a contour splits each spider in two; erasing that contour will reunite the spiders and have no other effect. So, to transform \( D' \) into \( D \), we erase the contour introduced into \( D \). Thus, \( D' \vdash D \). So, \( D' \models D \), by soundness.

A \( \beta \) diagram is an \( \alpha \) diagram in which each zone is either shaded or contains at least one spider. Any diagram \( D \) can be transformed into its \( \beta \) diagram \( D^\beta \) by repeated application of rule 6, splitting spiders, to turn it into an \( \alpha \) diagram and then repeated application of rule 7, rule of excluded middle. Figure 11 illustrates the transformation \( D \vdash D^\beta \) for a unitary diagram \( D \).

**Figure 11**

Theorem 11  For any diagram \( D, D^\beta \vdash D \).
Proof Any unitary component of \( D^\beta \) can be transformed into \( D \) by removing shading and extending spiders appropriately (i.e., by undoing the transformations to obtain that component). So, by rule 9, the rule of construction, \( D^\beta \vdash D \).

Theorem 12 Let \( D \) and \( D' \) be \( \beta \) unitary diagrams for which \( L(D') \subseteq L(D) \). Then the following three statements are equivalent.

(i) \( D \vdash D' \)

(ii) \( D \models D' \)

(iii) \[1\] \( \forall z' \in Z^*(D') \exists r \in R^*(D) \bullet z \subseteq_c r \) and 

\[2\] \( \forall r' \in R(D') \forall r \in R(D) \bullet r' \equiv_c r \quad \mid S(r) \mid \geq \mid S(r') \mid \).

Proof (i) (ii). By soundness.

(ii) (iii). We will prove the contrapositive: \( \neg([1] \land [2]) \rightarrow (D \not\models D') \). That is, \( \neg([1] \lor \neg [2]) \rightarrow (D \not\models D') \), which is equivalent to \( \neg[1] \rightarrow (D\not\models D') \land \neg [2] \rightarrow (D\models D') \).

(a) \( \neg[1] \rightarrow (D \not\models D') \). Assume \( \neg[1] \). That is, \( \exists z' \in Z^*(D') \forall r \in R^*(D) \bullet z \subseteq_c r \). Let \( z'_1 \) be such a \( z' \).

Let \( r_1 \in R(D) \) be such that \( r_1 \equiv_c z'_1 \). Then, by the assumption, \( r_1 \) is not entirely shaded and therefore contains spiders (as \( D \) is a \( \beta \) diagram). That is, \( |S(r_1)| > |S(r'_1)| \).

(b) \( \neg [2] \rightarrow (D \not\models D') \). Assume \( \neg [2] \). That is, \( \exists r' \in R(D') \exists r \in R(D) \bullet r' \equiv_c r \land |S(r')| < |S(r)| \).

Let \( r'_1 \in R(D') \) and \( r_1 \in R(D) \) be such that \( r'_1 \equiv_c r_1 \) and \( |S(r_1)| < |S(r'_1)| \). Let \( m = (\Psi, U) \) be such that \( m \models D \) and \( |\Psi(r_1)| = |S(r_1)| \). For any \( m' = (\Psi', U) \) compliant with \( D' \) we have, \( |\Psi'(r'_1)| \geq |S(r'_1)| > |S(r_1)| = |\Psi(r_1)| \), so \( m \) is not compliant with \( D' \).

(iii) (i). Let \( D \) and \( D' \) be \( \beta \) unitary diagrams for which \( L(D') \subseteq L(D) \) and assume \( [1] \) and \( [2] \). Then any region in \( D' \) has a corresponding equivalent region in \( D \). Let \( r \in R(D) \) be such that \( r \equiv_c \bigcup_{z \in Z^*(D')} z \), the region consisting of all and only shaded zones in \( D' \). Erase the shading in \( \bigcup_{z \in Z^*(D')} z - r \) to obtain \( D^1 \) so that \( \bigcup_{z \in Z^*(D')} z \equiv_c \bigcup_{z \in Z^*(D')} z \).

Remove contours from \( D^1 \) to obtain \( D^2 \) so that \( L(D^2) = L(D^2) \). Delete spiders in \( D^2 \) to obtain \( D^3 \) so that \( \forall r \in R(D) \forall r' \in R(D') \bullet r \equiv_c r' \quad |S(r)| = |S(r')| \). Then \( D^3 \equiv D' \).

Theorem 13 Let \( D, D_1, D_2, \ldots, D_n \) be \( \beta \) unitary diagrams for which \( L(D_1) \cup L(D_2) \cup \ldots \cup L(D_n) \subseteq L(D) \). Then 

\[ D \models D_1 \lor D_2 \lor \ldots \lor D_n \quad (D \models D_1 \lor D \models D_2 \lor \ldots \lor D \models D_n). \]

Proof We will prove the contrapositive. Assume \( \forall D_i \bullet D \not\models D_i \). So, by Theorem 12,

(i) \( \exists z_i \in Z^*(D_i) \exists r \in R^*(D) \bullet - (z_i \subseteq_c r) \) or (ii) \( \exists r_i \in R(D_i) \exists r \in R(D) \bullet r_i \equiv_c r \land |S(r)| < |S(r_i)| \).

Let \( m_1 = (\Psi, U) \) be compliant with \( D \) and be such that \( \forall z \in Z(D) \bullet |\Psi(z)| = |S(z)| \).
If condition (i) holds in $D_r$, then there will be a shaded zone $z_i$ in $D_i$ for which the corresponding equivalent region $r$ in $D_i$ contains a spider. For any model $m = (\Psi, U)$ compliant with $D_n, \Psi(z_i) = \emptyset$, and by Theorem 2, $\Psi(z_i) = \Psi(r)$ as $z_i \equiv_r r$. So $m_1$ is not compliant with $D_r$.

If condition (ii) holds in $D_n$, then there will be a region $r_i$ in $D_i$ for which the corresponding equivalent region $r$ in $D_i$ contains fewer spiders. That is, $r_i \equiv_r r \land |S(r)| < |S(r_i)|$. For any model $m = (\Psi, U)$ compliant with $D_n, |\Psi(r_i)| \geq |S(r_i)|$ and $\Psi(r_i) = \Psi(r)$. So $m_1$ is not compliant with $D_r$.

So, $\forall D_r \bullet m_1 \not\models D_i$, and hence, $m_1 \not\models D_1 - D_2 - \ldots - D_m$. That is, $D_1 - D_2 - \ldots - D_m$ is not compliant with $D_r$.

**Theorem 14** Let $D$ and $D'$ be compound diagrams for which each component is a $\beta$ unitary diagram. That is, $D = D_1^\beta - D_2^\beta - \ldots - D_k^\beta$ and $D' = D_1'^\beta - D_2'^\beta - \ldots - D_n'^\beta$. Assume further that $\forall D_1^\beta \bullet D_2^\beta \bullet L(D_n'^\beta) \subseteq L(D_n^\beta)$.

Then $D \models D'$.

**Proof** Assume $D \models D'$. Let $m = (\Psi, U)$ be any model such that $m \models D$. Then $m \models D'$. So $P_d(m) \models P_d'(m).$ That is, $\bigvee_{i=1}^k P_{D_1^\beta}(m) \models \bigvee_{i=1}^n P_{D_i^\beta}(m).$ By logical manipulation we have $\forall D_1^\beta \bullet P_{D_2^\beta}(m) \models \bigvee_{i=1}^n P_{D_i^\beta}(m).$ That is, $\forall D_1^\beta \bullet D_2^\beta \models D_1'^\beta - D_2'^\beta - \ldots - D_n'^\beta$. Hence, by Theorem 13, $\forall D_1^\beta \bullet D_2^\beta \models D_1^\beta \models D_1'^\beta$.

**Theorem 15** Completeness Theorem Let $\Delta$ be a multi-diagram and let $D'$ be a diagram. Then $\Delta \models D' \quad \Delta \vdash D'$.

**Proof** If $\Delta$ is inconsistent, then the result follows immediately by applying Rule 10. Assume that $\Delta$ is consistent and that $\Delta \models D'$. By Theorem 9, $\Delta^* \models \Delta$. So, by transitivity, $\Delta^* \models D'$. Introduce contours into each component of $D^*$ to produce $D^* = D_1^* - D_2^* - \ldots - D_m^*$, so that $\forall D_1^* \bullet L(D') \subseteq L(D^*)$. Then, by Theorem 10, $D^* \models D^*$. So, by transitivity, $D^*_\beta \models D'$. Transform $D^*_\beta$ and $D'$ into their $\beta$ diagrams, $D^*_{\beta^\beta}$ and $D_{\beta^\beta}$, respectively. By Theorem 11 and the soundness theorem, $D^*_{\beta^\beta} \models D^*$ and, by the soundness theorem, $D' \models D_{\beta^\beta}$. So, by transitivity, $D_{\beta^\beta} \models D'$. Since $\forall D_1^* \bullet D_{\beta^\beta} \bullet L(D_{\beta^\beta}) \subseteq L(D_{\beta^\beta})$, it follows from Theorem 14 that $\forall D_1^* \bullet D_{\beta^\beta} \models D'$. Hence, by Theorem 12, $\forall D_1^* \bullet D_{\beta^\beta} \models D_1^\beta$. So, by applying rule 8, $\forall D_1^* \bullet D_{\beta^\beta} \Rightarrow D_1^\beta$ and hence, by applying rule 9, $D_1^* \models D_1^\beta$. Now, $\Delta \models D^*$, $D^* \models D$ and, by Theorem 11, $D^\beta \models D'$, so, by transitivity, $\Delta \models D'$.

6. Conclusion and related work

We have given formal syntax and semantics and diagrammatic inference rules to simple spider diagrams. We have shown that the inference rules are sound and complete. In proving completeness, we have provided a proof strategy that should be extensible to most spider/constraint diagram systems and other similar systems based on Venn or Euler diagrams. Indeed, the proof can be adapted to give a simpler proof of the completeness of the Venn-II system than the one given by Shin.

We are in the process of proving soundness and completeness of other spider diagram systems using the same strategy as that introduced in this paper. Our longer term aim is to prove similar results for constraint diagrams, and to provide the necessary mathematical underpinning for the development of software tools to aid the reasoning process.
References